Mid-Semestral Exam
B.Math Algebra-IV
2015-2016

Time: 3 hrs
Max score: 100

Answer question (1) and any **four** from the rest.

(1) Prove or disprove.
(a) Let $F(\alpha)$ be an extension of a field F such that $[F(\alpha) : F]$ is odd. Then $F(\alpha) = F(\alpha^2)$.
(b) Every algebraic extension is a finite extension.
(c) If α is a real root of the polynomial $x^3 + 3x + 1$, then α cannot be constructed by ruler and compass.
(d) There exist finite fields which are not perfect.
(e) Every finite extension of a perfect field is separable. (4×5)

(2) Let $\alpha \in \mathbb{C}$ be a root of $f(x) = x^3 - x^2 - 4x - 1$.
(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$, clearly stating any result that you use.
(b) Prove that $-(1+\alpha)^{-1}$ is also a root of f.
(c) Prove or disprove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois.
(d) If $\zeta = e^{2\pi i/3}$, prove that f is irreducible over $\mathbb{Q}(\zeta)$. (5+4+6+5)

(3) (a) Let $f(x) \in F[x]$ be a monic polynomial of degree $n > 0$. Show that if K is the splitting field of $f(x)$ over F, then $[K : F]$ divides $n!$.
(b) Find the splitting field of $x^4 + 2$ over
(i) $\mathbb{Q}$ and (ii) $\mathbb{F}_3$. (8+6+6)

(4) (a) Let p be a prime. Prove that for each positive integer r, there is a unique finite field F, up to an isomorphism, such that $|F| = p^r$..
(b) If $\mathbb{F}_{p^r}$ denotes the unique finite field of order p^r, then show that $\mathbb{F}_{p^r}|\mathbb{F}_p$ is a Galois extension.
(c) Also show that the Galois group $\mathrm{Gal}(\mathbb{F}_{p^r}|\mathbb{F}_p)$ is the cyclic group of order r. (8+4+8)

(5) (a) Let K be a finite extension of $\mathbb{Q}$. Prove that there are only a finite number of roots of unity in K.
(b) Determine the automorphism group $\mathrm{Aut}(\mathbb{R}|\mathbb{Q})$. (8+12)

Please turn over

(6) Let $f(x) = x^3 - 5$ over $\mathbb{Q}$.
(a) Determine the splitting field of $f(x)$ over $\mathbb{Q}$, say L, and the degree of L over $\mathbb{Q}$.
(b) Show that the Galois group $Gal(L|\mathbb{Q})$ is isomorphic to the symmetric group S_3.
(c) Determine all non-trivial subgroups of the Galois group, and the corresponding fixed fields, over $\mathbb{Q}$. (6+6+8)

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